

General solution of classical master equation for reducible gauge theories

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Abstract

We give the general solution to the classical master equation $(S,S)=0$ for reducible gauge theories. To this aim, we construct a new coordinate system in the extended configuration space and transform the equation by changing variables. Then it can be solved by an iterative method.

1 Introduction

The classical master equation [1, 2, 3] arises in the Lagrangian approach to gauge theories within the BRST formalism [4, 5]. It reads

$$(S, S) = 0, \quad (1)$$

where S is an extended action, and $(.,.)$ is an antibracket (for a review of reducible gauge theories see refs. [6, 7] and references therein). The action S also satisfies certain boundary conditions. The general solution to this equation for irreducible gauge theories was described in [8, 9]. A similar equation arises in the Hamiltonian approach to the classical BRST charge. In both the Lagrangian and Hamiltonian formalisms an important role is played by the Koszul-Tate differential operator δ . The existence theorem for the BRST charge [10] is based on nilpotency and acyclicity of δ . By using the results of [10], an existence proof of solutions to (1) for arbitrary gauge theories was given in [11]. The authors of ref. [12] reviewed existence and uniqueness proofs for the extended action in the reducible case and obtained new ones.

In this paper we present the general solution to the master equation for reducible gauge theories. Our construction is based on a special representation of δ . We find the coordinates in the space of fields and antifields which

bring δ to a standard form. With respect to the new variables the master equation is simplified and can be solved by an iterative method.

A similar coordinate system was used for reducing the Koszul-Tate operator and computing cohomology groups of the BRST differential operator in irreducible gauge theories of the Yang-Mills type [13].

The paper is organized as follows. In section 2, we introduce notations and derive an auxiliary equation. In section 3, we construct new coordinates and transform the operator δ to a standard form. The general solution to the master equation is given in section 4. Some examples of reducible theories are discussed in section 5.

In what follows the Grassmann parity and ghost number of a function X are denoted by $\epsilon(X)$ and $\text{gh}(X)$, respectively.

2 The classical master equation

Let $S_0(\phi^{a_0})$ be an action depending on m_0 fields ϕ^{a_0} of Grassmann parity ϵ_{a_0} . The action is assumed to be gauge invariant

$$S_{0,a_0}(\phi^{b_0})R_{a_1}^{a_0}(\phi^{b_0}) = 0, \quad a_0, b_0 = 1, \dots, m_0, \quad a_1 = 1, \dots, m_1. \quad (2)$$

The set of generators $R_{a_1}^{a_0}$ forms an algebra,

$$R_{b_1, b_0}^{a_0} R_{a_1}^{b_0} - (-)^{\epsilon_{a_1} \epsilon_{b_1}} R_{a_1, b_0}^{a_0} R_{b_1}^{b_0} = -R_{c_1}^{a_0} F_{a_1 b_1}^{c_1} - S_{0, b_0} E_{a_1 b_1}^{a_0 b_0}.$$

Here ϵ_{a_1} is the Grassmann parity of the gauge parameter associated with the index a_1 .

We shall consider a reducible theory of L -th order. That is, there exist functions

$$R_{a_{k+1}}^{a_k}(\phi^{b_0}), \quad k = 0, \dots, L, \quad a_k = 1, \dots, m_k,$$

such that at each stage the R 's form a complete set,

$$R_{a_{k+1}}^{a_k} \lambda^{a_{k+1}} \approx 0 \Leftrightarrow \lambda^{a_{k+1}} \approx R_{a_{k+2}}^{a_{k+1}} \nu^{a_{k+2}}, \quad k = 0, \dots, L-1,$$

$$R_{a_{L+1}}^{a_L} \lambda^{a_{L+1}} \approx 0 \Leftrightarrow \lambda^{a_{L+1}} \approx 0,$$

$$R_{a_{k+1}}^{a_k} R_{a_{k+2}}^{a_{k+1}} = V_{a_{k+2}}^{a_k a_0} S_{0, a_0}, \quad k = 0, \dots, L-1. \quad (3)$$

Here $V_{a_2}^{a_0 b_0} = -(-1)^{\epsilon_{a_0} \epsilon_{b_0}} V_{a_2}^{b_0 a_0}$. The weak equality \approx means equality on the stationary surface

$$\Sigma: S_{0, a_0}(\phi^{b_0}) = 0.$$

The configuration space of the theory is extended by adding the ghost fields $C = (\phi^{a_1}, \phi^{a_2}, \dots, \phi^{a_{L+1}})$, and the antifields $\phi^* = (\phi_{a_0}^*, \phi_{a_1}^*, \dots, \phi_{a_{L+1}}^*)$,

$$\text{gh}(\phi^{a_k}) = k, \quad \text{gh}(\phi_{a_k}^*) = -\text{gh}(\phi^{a_k}) - 1, \quad \epsilon(\phi_{a_k}^*) = \epsilon(\phi^{a_k}) + 1.$$

We shall seek the extended action $S = S(\phi, \phi^*)$, $\phi = (\phi^{a_0}, C)$, in the form of expansions in power series of the ghost fields. The antibracket is defined by

$$(X, Y) = \sum_{k=0}^{L+1} \left(\frac{\delta X}{\delta \phi^{a_k}} \frac{\delta Y}{\delta \phi_{a_k}^*} - (-)^{(\epsilon(X)+1)(\epsilon(Y)+1)} (X \leftrightarrow Y) \right).$$

Derivatives with respect to antifields are always understood as left, while with respect to fields as right.

Eq. (1) is supplied by the following conditions

$$\epsilon(S) = 0, \quad \text{gh}(S) = 0, \quad (4)$$

$$S|_{C=\phi^*=0} = S_0, \quad \left. \frac{\delta^2 S}{\delta \phi_{a_{k-1}}^* \delta \phi^{a_k}} \right|_{C=\phi^*=0} = R_{a_k}^{a_{k-1}}, \quad k = 1, \dots, L+1.$$

We assume that S is a local functional.

One can write

$$S = S_0 + S_1 + K, \quad K = \sum_{n \geq 2} S_n, \quad S_n \sim C^n, \quad (5)$$

$$S_1 = \sum_{k=1}^{L+1} \left(\phi_{a_{k-1}}^* R_{a_k}^{a_{k-1}} + M_{a_k} \right) \phi^{a_k}, \quad (6)$$

where $M_{a_k} = M_{a_k}(\phi^{a_0}, \phi_{a_1}^*, \dots, \phi_{a_{k-2}}^*)$. Eq. (4) implies that

$$M_{a_k}|_{\phi^*=0} = 0, \quad K|_{\phi^*=0} = 0.$$

Let \mathcal{V} be the space of the local functionals depending on (ϕ, ϕ^*) which vanish on Σ at $\phi^* = 0$. It is easily verified that $S_n, n \geq 1$, as well as (S, S) , belong to \mathcal{V} .

Substituting (5) in (1) one obtains

$$\delta S_1 = 0, \quad (7)$$

$$\delta K = D, \quad (8)$$

where

$$\delta = S_{0,a_0} \frac{\delta}{\delta \phi_{a_0}^*} + \sum_{k=1}^{L+1} \left(\phi_{a_{k-1}}^* R_{a_k}^{a_{k-1}} + M_{a_k} \right) \frac{\delta}{\delta \phi_{a_k}^*}, \quad (9)$$

$$-D = B + AK + \frac{1}{2}(K, K), \quad B = S_{1,a_0} R_{a_1}^{a_0} \phi^{a_1},$$

and the operator A is defined by

$$AX = S_{1,a_0} \frac{\delta X}{\delta \phi_{a_0}^*} + (-1)^{\epsilon(X)} \sum_{k=0}^{L+1} \frac{\delta X}{\delta \phi^{a_k}} \frac{\delta S_1}{\delta \phi_{a_k}^*}.$$

Eq. (7) is equivalent to

$$\delta^2 = 0. \quad (10)$$

Assume that (7) holds. Then $(S, S) = G$, where G is the difference between the left-hand and right-hand sides of (8),

$$G = \delta K + B + AK + \frac{1}{2}(K, K).$$

From the Jacobi identity $(S, (S, S)) = 0$ it follows that $(S, G) = 0$, or equivalently

$$\delta G + AG + (K, G) = 0. \quad (11)$$

3 Reduction of δ

For $k = L$ eq. (3) reads

$$R_{a'_L}^{a_{L-1}} R_{a_{L+1}}^{a'_L} + R_{a''_L}^{a_{L-1}} R_{a_{L+1}}^{a''_L} \approx 0, \quad (12)$$

where a'_L, a''_L are index sets, such that $a'_L \cup a''_L = a_L$, $|a'_L| = |a_{L+1}|$, and $\text{rank } R_{a_{L+1}}^{a'_L} \Big|_{\Sigma} = |a_{L+1}|$.¹ It follows from (12) that

$$\text{rank } R_{a_L}^{a_{L-1}} \Big|_{\Sigma} = \text{rank } R_{a'_L}^{a_{L-1}} \Big|_{\Sigma} = |a''_L|.$$

One can split the index set a_{L-1} as $a_{L-1} = a'_{L-1} \cup a''_{L-1}$, such that $|a'_{L-1}| = |a''_L|$ and $\text{rank } R_{a''_L}^{a'_{L-1}} \Big|_{\Sigma} = |a''_L|$. For $k = L - 1$ eq. (3) implies

$$R_{a'_{L-1}}^{a_{L-2}} R_{a''_L}^{a'_{L-1}} + R_{a''_{L-1}}^{a_{L-2}} R_{a''_L}^{a''_{L-1}} \approx 0.$$

From this it follows that

$$\text{rank } R_{a_{L-1}}^{a_{L-2}} \Big|_{\Sigma} = \text{rank } R_{a''_{L-1}}^{a_{L-2}} \Big|_{\Sigma} = |a''_{L-1}|.$$

Using induction on k , one can obtain a set of matrices $R_{a''_k}^{a_{k-1}}$, $k = 1, \dots, L + 1$, satisfying

$$\text{rank } R_{a''_k}^{a_{k-1}} \Big|_{\Sigma} = \text{rank } R_{a_k}^{a_{k-1}} \Big|_{\Sigma} = |a''_k|,$$

¹For an index set $i = \{i_1, i_2, \dots, i_n\}$, $|i| = n$.

and a set of nonsingular matrices $R_{a''_k}^{a'_{k-1}}$, $k = 1, \dots, L+1$, where $a'_k \cup a''_k = a_k$.

Eq. (2) implies

$$S_{0,b_0 a'_0} R_{a''_1}^{a'_0} + S_{0,b_0 a''_0} R_{a''_1}^{a''_0} \approx 0,$$

and therefore

$$\text{rank } S_{0,b_0 a_0} \Big|_{\Sigma} = \text{rank } S_{0,b_0 a''_0} \Big|_{\Sigma} = |a''_0|.$$

For $a''_{k+1} \subset a_{k+1}$, $k = 0, \dots, L-1$, we define an embedding $f(a''_{k+1}) = a''_{k+1} \subset a_k$, $f(a''_{L+1}) = a_{L+1} \subset a_L$. Thus, for example, $(\phi_{f(a''_{k+1})}^*) \subset (\phi_{a_k}^*)$. Let α_k , $k = 0, \dots, L$, be defined by $a_k = f(a''_{k+1}) \cup \alpha_k$. One can write $\alpha_k = g(a''_k)$ for some function g , since $|a''_k| = |\alpha_k|$.

Lemma. The nilpotent operator δ is reducible to the form

$$\delta = \phi'_{a''_0} \frac{\delta}{\delta \phi_{g(a''_0)}^{*'}} + \sum_{k=1}^{L+1} \phi_{f(a''_k)}^{*'} \frac{\delta}{\delta \phi_{g(a''_k)}^{*'}},$$

by the change of variables: $(\phi^{a_0}, \phi^*) \rightarrow (\phi'_{a_0}, \phi^{*'})$,

$$\phi'_{a''_0} = S_{0,a''_0}, \quad \phi'_{a'_0} = \phi^{a'_0},$$

$$\phi_{f(a''_{k+1})}^{*'} = \phi_{a_k}^* R_{a''_{k+1}}^{a_k} + M_{a''_{k+1}}, \quad \phi_{\alpha_k}^{*'} = \phi_{g^{(-1)}(\alpha_k)}^*, \quad (13)$$

$$\phi_{a_{L+1}}^{*'} = \phi_{a_{L+1}}^*,$$

where $k = 0, \dots, L$, $g(a''_{L+1}) = a_{L+1}$.

To prove this statement we first observe that the matrices $(\tilde{S}_{a_0 b_0}) = (S_{0,a''_0 b''_0}, \delta_{a'_0 b'_0})$, $(\tilde{R}_{a_k}^{b_k}) = (R_{f(a''_{k+1})}^{b_k}, \delta_{\alpha_k}^{b_k})$, $k = 0, \dots, L$, are invertible, and therefore transformation (13) is nonsingular. For $0 \leq s \leq L$

$$\frac{\delta}{\delta \phi_{a'_s}^*} = \sum_{k=0}^L \frac{\delta(\delta \phi_{a''_{k+1}}^*)}{\delta \phi_{a'_s}^*} \frac{\delta}{\delta \phi_{f(a''_{k+1})}^{*'}}, \quad \frac{\delta}{\delta \phi_{a''_s}^*} = \sum_{k=0}^L \frac{\delta(\delta \phi_{a''_{k+1}}^*)}{\delta \phi_{a''_s}^*} \frac{\delta}{\delta \phi_{f(a''_{k+1})}^{*'}} + \frac{\delta}{\delta \phi_{g(a''_s)}^{*'}}.$$

Substituting this in (9), we get

$$\delta = \sum_{k=0}^L \left(\delta^2 \phi_{a''_{k+1}}^* \frac{\delta}{\delta \phi_{f(a''_{k+1})}^{*'}} + \delta \phi_{a''_k}^* \frac{\delta}{\delta \phi_{g(a''_k)}^{*'}} \right) + \delta \phi_{a_{L+1}}^* \frac{\delta}{\delta \phi_{a_{L+1}}^{*'}}.$$

The result then follows from (10) and (13).

Eqs. (13) are solvable with respect to the original variables and can be represented as

$$\phi^{a_0} = \phi^{a_0}(\phi'_{b_0}), \quad \phi_{a_k}^* = \phi_{a_k}^*(\phi'_{a_0}, \phi_{a_0}^{*'}, \dots, \phi_{a_k}^{*'}), \quad k = 0, \dots, L+1.$$

Here we have used the fact that the $\phi_{a_k}^*$ depends only on the functions $\phi_{a_s}^{*'}$ with $s \leq k$. Assume that the functions $\phi^{a_0}(\phi_{b_0}')$ have been constructed. Then from (13)

$$\phi_{a_k}^* = (\phi_{f(a_{k+1}'')}^{*'} - \phi_{g(a_k'')}^{*'} R_{a_{k+1}''}^{a_k''} - M_{a_{k+1}''}') (R'^{(-1)})_{a_k}^{a_{k+1}''}, \quad \phi_{a_k}^{*'} = \phi_{g(a_k'')}^{*'},$$

$$\phi_{a_{L+1}}^* = \phi_{a_{L+1}}^{*'}, \quad (14)$$

where $k = 0, \dots, L$,

$$R_{a_{k+1}}^{a_k}(\phi_{a_0}') = R_{a_{k+1}}^{a_k}(\phi^{a_0}), \quad M_{a_k}'(\phi_{a_0}', \phi_{a_0}^{*'}, \dots, \phi_{a_{k-2}}^{*'}) = M_{a_k}(\phi^{a_0}, \phi_{a_0}^*, \dots, \phi_{a_{k-2}}^*).$$

With respect to the new coordinate system the condition $X \in \mathcal{V}$ implies

$$X|_{\phi_{a_0}'' = \phi^{*'} = 0} = 0.$$

For $X_i'(\phi_{a_0}', \phi^{*'}, C)$, $i = 1, 2$, we define

$$(X_1', X_2')' = (X_1, X_2),$$

where

$$X_i(\phi^{a_0}, \phi^*, C) = X_i'(\phi_{a_0}', \phi^{*'}, C).$$

Notice that the variables ϕ_{a_0}' , $\phi_{g(a_0'')}^{*'}', \phi_{f(a_1'')}^{*'}', \phi_{g(a_1'')}^{*'}', \dots, \phi_{f(a_{L+1}'')}^{*'}', \phi_{g(a_{L+1}'')}^{*'}'$ are independent.

4 The extended action

Solution of the equation $\delta^2 = 0$. Eq. (10) is equivalent to the recurrent relations $M_{a_1} = 0$,

$$\delta M_{a_k} = -(\phi_{a_{k-2}}^* R_{a_{k-1}}^{a_{k-2}} + M_{a_{k-1}}) R_{a_k}^{a_{k-1}}, \quad k = 2, \dots, L+1. \quad (15)$$

One can replace δ by δ_k ,

$$\delta_k = S_{0, a_0} \frac{\delta}{\delta \phi_{a_0}^*} + \sum_{s=1}^{k-2} \left(\phi_{a_{s-1}}^* R_{a_s}^{a_{s-1}} + M_{a_s} \right) \frac{\delta}{\delta \phi_{a_s}^*},$$

since M_{a_k} does not depend on $\phi_{a_s}^*$, $s > k-2$. The operator δ_k and right-hand side of (15) only involves the functions M_{a_s} with $s < k$.

Assume that the functions M_{a_s} , $s < k$, have been constructed. Changing variables in (15) $(\phi^{a_0}, \phi_{a_0}^*, \dots, \phi_{a_{k-2}}^*) \rightarrow (\phi_{a_0}', \phi_{a_0}^{*'}, \dots, \phi_{a_{k-2}}^{*'})$, we get

$$\delta_k M_{a_k}' = D_{a_k}', \quad (16)$$

where

$$\delta_k = \phi'_{a_0} \frac{\delta}{\delta \phi_{g(a_0)}^{*'}} + \sum_{s=1}^{k-2} \phi_{f(a_s)}^{*'} \frac{\delta}{\delta \phi_{g(a_s)}^{*'}}, \quad D'_{a_k} = -(\phi_{a_{k-2}}^* R_{a_{k-1}}'^{a_{k-2}} + M_{a_{k-1}}') R_{a_k}'^{a_{k-1}},$$

$$\phi_{a_{k-2}}^* = \phi_{a_{k-2}}^*(\phi'_{a_0}, \phi_{a_0}^{*'}, \dots, \phi_{a_{k-2}}^{*'}).$$

Let n_k be the counting operator

$$n_k = \phi'_{a_0} \frac{\delta}{\delta \phi_{a_0}'} + \phi_{g(a_0)}^{*'} \frac{\delta}{\delta \phi_{g(a_0)}^{*'}} + \sum_{s=1}^{k-2} \left(\phi_{f(a_s)}^{*'} \frac{\delta}{\delta \phi_{f(a_s)}^{*'}} + \phi_{g(a_s)}^{*'} \frac{\delta}{\delta \phi_{g(a_s)}^{*'}} \right),$$

and let

$$\sigma_k = \phi_{g(a_0)}^{*'} \frac{\delta}{\delta \phi_{a_0}'} + \sum_{s=1}^{k-2} \phi_{g(a_s)}^{*'} \frac{\delta}{\delta \phi_{f(a_s)}^{*'}}.$$

One can directly verify that

$$\delta_k^2 = \sigma_k^2 = 0, \quad \delta_k \sigma_k + \sigma_k \delta_k = n_k, \quad n_k \delta_k = \delta_k n_k, \quad n_k \sigma_k = \sigma_k n_k. \quad (17)$$

Let \mathcal{V}_k , $2 \leq k \leq L+3$, be the subspace of \mathcal{V} which consists of the functionals depending only on $(\phi'_{a_0}, \phi_{a_0}^{*'}, \dots, \phi_{a_{k-2}}^{*'})$. Notice that $\mathcal{V}_{L+3} = \mathcal{V}$. The space \mathcal{V}_k splits as

$$\mathcal{V}_k = \mathcal{V}_k^{(0)} \oplus \tilde{\mathcal{V}}_k, \quad \tilde{\mathcal{V}}_k = \mathcal{V}_k^{(1)} \oplus \mathcal{V}_k^{(2)} \oplus \dots,$$

with $n_k X = nX$ for $X \in \mathcal{V}_k^{(n)}$. It is clear that

$$\mathcal{V}_k^{(0)} = \{\Phi \in \mathcal{V}_k \mid \Phi = \Phi(\phi'_{a_0}, \phi_{f(a_{k-1})}^{*'})\}, \quad k \neq L+3, \quad \mathcal{V}_{L+3}^{(0)} = 0. \quad (18)$$

We define $n_k^+ : \mathcal{V}_k \rightarrow \mathcal{V}_k$ by

$$n_k^+ X = \begin{cases} n_k^{(-1)} X, & X \in \tilde{\mathcal{V}}_k; \\ 0, & X \in \mathcal{V}_k^{(0)}. \end{cases}$$

where $n_k^{(-1)} : \tilde{\mathcal{V}}_k \rightarrow \tilde{\mathcal{V}}_k$ is given by

$$n_k^{(-1)} X = \frac{1}{n} X, \quad X \in \mathcal{V}_k^{(n)}, \quad n > 0.$$

Then $\delta_k^+ = \sigma_k n_k^+$ is a generalized inverse of δ_k :

$$\delta_k \delta_k^+ \delta_k = \delta_k, \quad \delta_k^+ \delta_k \delta_k^+ = \delta_k^+. \quad (19)$$

We shall denote $\delta^+ = \delta_{L+3}^+$.

Eq. (3) takes the form

$$R'_{a_{k+1}} R'^{a_{k+1}}_{a_{k+2}} = V'^{a_k a'_0}_{a_{k+2}} S'_{a'_0} + V'^{a_k a''_0}_{a_{k+2}} \phi'_{a''_0}, \quad k = 0, \dots, L-1,$$

where

$$V'^{a_0 b_0}_{a_2}(\phi'_{a_0}) = V^{a_0 b_0}_{a_2}(\phi^{a_0}), \quad S'_{b'_0}(\phi'_{a_0}) = S_{0, b'_0}(\phi^{a_0}).$$

It follows from (2) that

$$S'_{a'_0} = -\phi'_{a'_0} U'^{a''_0}_{a'_0}, \quad U'^{a''_0}_{a'_0} = R'^{a''_0}_{a'_1} (R'^{(-1)})^{a''_1}_{a'_0},$$

and hence

$$R'^{a_k}_{a_{k+1}} R'^{a_{k+1}}_{a_{k+2}} \in \tilde{\mathcal{V}}_2. \quad (20)$$

We assume that $M'_{a_s} \in \tilde{\mathcal{V}}_s$, $s < k$. Then, by (20), $D'_{a_k} \in \tilde{\mathcal{V}}_k$. One can directly verify that $\delta_k D'_{a_k} = 0$, or equivalently $\delta_k \delta_k^+ D'_{a_k} = D'_{a_k}$. Then the general solution to (16) for $2 \leq k \leq L$ is given by

$$M'_{a_k} = Y'_{a_k} + \delta_k^+ D'_{a_k}, \quad (21)$$

where $Y'_{a_k} \in \tilde{\mathcal{V}}_k$ is an arbitrary cocycle, $\delta_k Y'_{a_k} = 0$, subject only to the restrictions

$$\epsilon(Y'_{a_k}) = \epsilon(M'_{a_k}), \quad \text{gh}(Y'_{a_k}) = \text{gh}(M'_{a_k}). \quad (22)$$

By its construction, $M'_{a_k} \in \tilde{\mathcal{V}}_k$. The function $M'_{a_{L+1}}$ is given by (21), (22), where the cocycle $Y'_{a_{L+1}}$ belongs to \mathcal{V}_{L+1} .

Higher orders. Consider eq. (8). Changing variables $(\phi^{a_0}, \phi^*) \rightarrow (\phi'_{a_0}, \phi'^*)$, we get

$$\delta K' = D', \quad (23)$$

where

$$-D' = B' + AK' + \frac{1}{2}(K', K')$$

$$B'(\phi', \phi'^*, C) = B(\phi, \phi^*), \quad K'(\phi', \phi'^*, C) = K(\phi, \phi^*).$$

Applying $\delta\delta^+$ to eq. (23) and using (19) we have

$$\delta K' = \delta\delta^+ D',$$

from which it follows that

$$K' = Y' + \delta^+ D', \quad (24)$$

$$Y' \in \mathcal{V}, \quad \delta Y' = 0, \quad \epsilon(Y') = 0, \quad \text{gh}(Y) = 0. \quad (25)$$

Let $\langle ., . \rangle : \mathcal{V}^2 \rightarrow \mathcal{V}$ be defined by

$$\langle X'_1, X'_2 \rangle = -\frac{1}{2}(I + \delta^+ A)^{-1} \delta^+ ((X'_1, X'_2)' + (X'_2, X'_1)'),$$

where I is the identity map, and

$$(I + \delta^+ A)^{-1} = \sum_{m \geq 0} (-1)^m (\delta^+ A)^m.$$

One can rewrite (24) as

$$K' = K'_0 + \frac{1}{2} \langle K', K' \rangle, \quad (26)$$

where

$$K'_0 = (I + \delta^+ A)^{-1} (Y' - \delta^+ B'),$$

Eq. (26) can be iteratively solved as:

$$K' = K'_0 + \frac{1}{2} \langle K'_0, K'_0 \rangle + \dots \quad (27)$$

To prove that the solution to eq. (24) satisfies (23) we shall use the approach of ref. [14]. From (17) and the second relation of (18) it follows that for any $X \in \mathcal{V}$,

$$X = \delta^+ \delta X + \delta \delta^+ X. \quad (28)$$

Changing variables in (11) from (ϕ^{a_0}, ϕ^*) to (ϕ'_{a_0}, ϕ'^*) , we get

$$\delta G' + A G' + (K', G')' = 0, \quad (29)$$

where

$$G' = \delta K' + B' + A K' + \frac{1}{2} (K', K')'. \quad (30)$$

Consider eq. (29) and the condition

$$\delta^+ G' = 0, \quad (31)$$

where K' is the solution to (24). Applying δ^+ to eq. (29) and using (31) we get

$$G' = -\delta^+ (A G' + (K', G')'), \quad (32)$$

since $G' \in \mathcal{V}$. From (32) it follows that $G' = 0$.

The solution to (24) satisfies the condition

$$\delta^+ K' = \delta^+ Y',$$

since $(\delta^+)^2 = 0$. By using (28), we have

$$K' = \delta^+ \delta K' + \delta \delta^+ Y'.$$

From this, (28) and (25) it follows that

$$K' = \delta^+ \delta K' + Y'. \quad (33)$$

To check (31), we have

$$\delta^+ G' = \delta^+ \delta K' + \delta^+ (B' + AK' + \frac{1}{2}(K', K')'),$$

and therefore by (33) and (24), $\delta^+ G' = 0$.

Changing variables in (21) and (27) from (ϕ'_{a_0}, ϕ'^*) to (ϕ^{a_0}, ϕ^*) , one can obtain $S_1(\phi, \phi^*)$ and $K(\phi, \phi^*)$. The construction of the inverse transformation may be problematic in a field theory. It should be noted, however, that the extended action can be computed by using formal expressions (14), because the result does not depend on the choice of an auxiliary coordinate system.

5 Reducible abelian gauge theories

In this section we discuss a class of reducible bosonic theories including, in particular, the antisymmetric tensor and abelian p -form gauge theories.

It follows from (26) that if $\delta^+ B' = Y'$ then $K = 0$, and therefore

$$S = S_0 + S_1. \quad (34)$$

Let us consider the case of $B' = 0$, $Y' = 0$. The equation $B = 0$ is equivalent to the relations

$$R_{b_1, b_0}^{a_0} R_{a_1}^{b_0} - R_{a_1, b_0}^{a_0} R_{b_1}^{b_0} = 0, \quad (35)$$

$$R_{a_k, b_0}^{a_{k-1}} R_{a_1}^{b_0} = 0, \quad k = 2, \dots, L+1, \quad (36)$$

$$M_{a_k, a_0} R_{a_1}^{a_0} = 0. \quad (37)$$

Eq. (35) tells us that the gauge algebra is abelian.

Let us consider the case of reducible abelian gauge theories of the first order. The reducibility identities read

$$R_{a_1}^{a_0} R_{a_2}^{a_1} = V_{a_2}^{a_0 b_0} S_{0, b_0}, \quad V_{a_2}^{a_0 b_0} = -V_{a_2}^{b_0 a_0}. \quad (38)$$

We assume that $V_{a_2}^{a_0 b_0}$ are constants, and (36) is satisfied.

It is easily verified that for $k = 2$ the general solution to (15) is given by

$$M_{a_2} = Y_{a_2} + \frac{1}{2} \phi_{a_0}^* V_{a_2}^{a_0 b_0} \phi_{b_0}^*, \quad (39)$$

where $Y_{a_2} = Y_{a_2}(\phi^{a_0}, \phi_{a_0}^*)$ is a cocycle. If $Y_{a_2, a_0} R_{a_1}^{a_0} = 0$, then M_{a_2} satisfies (37). We can set $Y_{a_2} = 0$. Then

$$S = S_0 + \phi_{a_0}^* R_{a_1}^{a_0} \phi^{a_1} + (\phi_{a_1}^* R_{a_2}^{a_1} + \frac{1}{2} \phi_{a_0}^* V_{a_2}^{a_0 b_0} \phi_{b_0}^*) \phi^{a_2}.$$

An example of such a theory is the antisymmetric tensor gauge theory [15] whose dynamics is described by the action

$$S_0 = \int d^4x \left(\frac{1}{2} A_\mu^a A^{a\mu} - \frac{1}{2} B_{\mu\nu}^a F^{a\mu\nu} \right), \quad (40)$$

where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f_{bc}^a A_\mu^b A_\nu^c$, f_{bc}^a are the structure constants of a semi-simple compact Lie algebra.

The action is invariant under the gauge transformations

$$\delta B^{a\mu\nu} = R_{b\lambda}^{a\mu\nu} \xi_\sigma^b, \quad \delta A_\mu^a = 0, \quad (41)$$

where

$$R_{b\lambda}^{a\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} \eta_{\sigma\lambda} \nabla_{\rho b}^a, \quad \nabla_{\mu b}^a = \partial_\mu \delta_b^a - f_{cb}^a A_\mu^c,$$

$\eta_{\sigma\lambda} = \text{diag}(-1, 1, 1, 1)$. It is easily seen that the gauge algebra is abelian. The equations of motion are

$$\frac{\delta S_0}{\delta B^{a\mu\nu}} = -F_{a\mu\nu} = 0, \quad \frac{\delta S_0}{\delta A^{a\mu}} = A_{a\mu} + \nabla_b^{\nu a} B_{\nu\mu}^b = 0. \quad (42)$$

This theory is on-shell first stage reducible, since

$$R_{b\lambda}^{c\mu\nu} R_a^{b\lambda} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} f_{ab}^c F_{\rho\sigma}^b, \quad (43)$$

and $R_a^{b\lambda} = \nabla_a^{\lambda b}$ are independent. It is easily seen that (36) holds.

We have

$$\phi^{a_0} = (B_{\mu\nu}^a(x), A_\lambda^b(y)), \quad a_0 = (a, \mu, \nu, x) \cup (b, \lambda, y), \quad \mu < \nu,$$

$$a'_0 = (a, i, j, x), \quad a''_0 = (a, k, 3, x) \cup (b, \lambda, y), \quad 0 \leq i, j, k \leq 2, \quad i < j,$$

$$\phi^{a_1} = c_\mu^a(x), \quad a_1 = (a, \mu, x), \quad a'_1 = (a, 3, x), \quad a''_1 = (a, i, x),$$

$$\phi^{a_2} = c^a(x), \quad a_2 = (a, x),$$

$$\phi_{a_0}^* = (B_{\mu\nu}^{*a}(x), A_\lambda^{*b}(y)), \quad \phi_{a_1}^* = c_\mu^{*a}(x), \quad \phi_{a_2}^* = c^{*a}(x).$$

Let $\nabla^{(-1)a}_b(x, y)$ denote an inverse of $\nabla_b^a \delta(x - y)$, $\nabla_b^a = \nabla_{3b}^a$,

$$\nabla_c^a(x) \nabla^{(-1)c}_b(x, y) = \delta_b^a \delta(x - y).$$

We impose the boundary conditions that all the fields and antifields vanish at $x_3 \rightarrow -\infty$. Then the inverse $\nabla^{(-1)a}_b(x, y)$ is unique. It can be obtained by iterating the equation

$$\nabla^{(-1)a}_b(x, y) = \delta_b^a (\delta^3 \theta)(x - y) - (\delta^3 \theta H_c^a \nabla^{(-1)c}_b)(x, y),$$

where

$$(\delta^3 \theta)(x) = \delta(x_0) \delta(x_1) \delta(x_2) \theta(x_3), \quad H_b^a(x, y) = f_{cb}^a A_3^c(x) \delta(x - y),$$

$\theta(x_3) = 1$ if $x_3 \geq 0$ and $\theta(x_3) = 0$ otherwise.

The operator

$$R_{a_2}^{a'_1} = R_a^{b_3} = \nabla_a^b$$

is invertible. Here and in what follows we omit space-time indexes. It is easily verified that

$$R_{a'_1}^{a'_0} = R_{b_k}^{a_{ij}} = \varepsilon^{lij} \eta_{lk} \nabla_b^a, \quad i < j,$$

is also invertible.

In accordance with (13), (42) the substitution $(B_{\mu\nu}^a, A_\lambda^b) \rightarrow (B_{\mu\nu}'^a, A_\lambda'^b)$ looks like

$$B_{i3}'^a = -F_{i3}^a, \quad A_\nu'^a = A_\nu^a + \nabla_b^{\mu a} B_{\mu\nu}^b, \quad B_{ij}'^a = B_{ij}^a.$$

Solving these equations with respect to $(B_{\mu\nu}^a, A_\lambda^b)$, we get

$$B_{ij}^a = B_{ij}'^a, \quad A_3^a = A_3'^a, \quad A_i^a = \nabla'^{(-1)a}_b (\partial_i A_3'^b + B_{i3}'^b),$$

$$B_{j3}^a = \nabla'^{(-1)a}_b (A_j^b - A_j'^b + \nabla_c'^{ib} B_{ij}'^c).$$

Here

$$A_j^a = A_j^a(A_3'^b, B_{i3}'^c), \quad \nabla_{\mu b}'^a (A_\nu'^c) = \nabla_{\mu b}^a (A_\nu^c).$$

Let us now consider the substitution $(B_{\mu\nu}^{*a}, A_\lambda^{*b}) \rightarrow (B_{\mu\nu}^{*a'}, A_\lambda^{*b'}) :$

$$B_{f(a,i)}^{*a'} = B_{b\mu\nu}^* \varepsilon^{\mu\nu\lambda j} \eta_{ji} \nabla_{\lambda a}^b, \quad B_{j3}^{*a'} = B_{j3}^{*a}, \quad A_\mu^{*a'} = A_\mu^{*a}, \quad (44)$$

where

$$f^{(-1)}(a, i, j) = (a, |\varepsilon^{ij0}| + \sum_{k>0} |\varepsilon^{ijk} k|), \quad i < j.$$

From (44) it follows that

$$B_{j3}^{*a} = B_{j3}^{*la}, \quad A_\mu^{*a} = A_\mu^{*la},$$

$$B_{aij}^* = (B_{ci3}^{*l} \nabla_{jb}^c - B_{cj3}^{*l} \nabla_{ib}^c - \frac{1}{2} \varepsilon_{lij} \eta^{lk} B_{f(b,k)}^{*l}) \nabla_a^{(-1)b}.$$

Let us derive an expression for $M_{a_2} = M_a$ from (21). By using (43), we get

$$\delta_2^+ D'_a = n_2^{(-1)} N'_a, \quad N'_a = -\frac{1}{4} \sigma_2 (\tilde{B}_{c\mu\nu}^* \varepsilon^{\mu\nu\rho\sigma} f_{ab}^c F_{\rho\sigma}^{*b}), \quad (45)$$

where

$$\tilde{B}_{a\mu\nu}^* = B_{a\mu\nu}^*(B', A', B^{*l}, A^{*l}), \quad F_{\mu\nu}^{*a}(A') = F_{\mu\nu}^a(A), \quad (46)$$

$B' = (B_{\mu\nu}^{*a})$, $A' = (A_\mu^{*a})$, $B^{*l} = (B_{\mu\nu}^{*la})$, $A^{*l} = (A_\mu^{*la})$. The operators n_2, σ_2, δ_2 are given by

$$n_2 = B_{i3}^{*la} \frac{\delta}{\delta B_{i3}^{*la}} + A_\mu^{*la} \frac{\delta}{\delta A_\mu^{*la}} + B_{i3}^{*la} \frac{\delta}{\delta B_{i3}^{*la}} + A_\mu^{*la} \frac{\delta}{\delta A_\mu^{*la}},$$

$$\sigma_2 = B_{i3}^{*la} \frac{\delta}{\delta B_{i3}^{*la}} + A_\mu^{*la} \frac{\delta}{\delta A_\mu^{*la}}, \quad \delta_2 = B_{i3}^{*la} \frac{\delta}{\delta B_{i3}^{*la}} + A_\mu^{*la} \frac{\delta}{\delta A_\mu^{*la}}.$$

One can write

$$N'_a = -\frac{1}{8} n_2 (\tilde{B}_{c\mu\nu}^* \varepsilon^{\mu\nu\rho\sigma} f_{ab}^c \tilde{B}_{\rho\sigma}^{*b}) + W'_a,$$

where

$$W'_a = \frac{1}{8} n_2 (\tilde{B}_{c\mu\nu}^* \varepsilon^{\mu\nu\rho\sigma} f_{ab}^c \tilde{B}_{\rho\sigma}^{*b}) - \frac{1}{4} \sigma_2 (\tilde{B}_{c\mu\nu}^* \varepsilon^{\mu\nu\rho\sigma} f_{ab}^c F_{\rho\sigma}^{*b}).$$

Then it follows from (21) and (45) that

$$M'_a = \tilde{Y}'_a - \frac{1}{8} \tilde{B}_{c\mu\nu}^* \varepsilon^{\mu\nu\rho\sigma} f_{ab}^c \tilde{B}_{\rho\sigma}^{*b}, \quad (47)$$

where

$$\tilde{Y}'_a = Y'_a + Z'_a, \quad Z'_a = n_2^{(-1)} W'_a.$$

Changing variables in (47) from (B', A', B^{*l}, A^{*l}) to (B, A, B^*, A^*) , we get

$$M_a = Y_a - \frac{1}{8} B_{c\mu\nu}^* \varepsilon^{\mu\nu\rho\sigma} f_{ab}^c B_{\rho\sigma}^{*b}, \quad (48)$$

where

$$Y_a(B, A, B^*, A^*) = \tilde{Y}'_a(B', A', B^{*'}, A^{*'}).$$

It remains to check that $\delta Z'_a = 0$. By the definition of $\delta B_{\mu\nu}^{*a}$ and (46),

$$\delta_2 \tilde{B}_{\mu\nu}^{*a} = -F_{\mu\nu}^{'a},$$

and therefore by (17),

$$Z'_a = \delta_2 \delta_2^+ X'_a = \delta \delta^+ X'_a,$$

where

$$X'_a = \frac{1}{8} \tilde{B}_{c\mu\nu}^* \varepsilon^{\mu\nu\rho\sigma} f_{ab}^c \tilde{B}_{\rho\sigma}^{*b}.$$

Substituting (48) with $Y_a = 0$ in (34), we get the extended action for (40)

$$S = S_0 + \int d^4x \left(\frac{1}{2} B_{a\mu\nu}^* \varepsilon^{\mu\nu\rho\lambda} \nabla_{\rho b}^a c_\lambda^b + c_{\mu a}^* \nabla_b^{\mu a} c^b + \frac{1}{8} B_{c\mu\nu}^* \varepsilon^{\mu\nu\rho\sigma} f_{ab}^c B_{\rho\sigma}^{*b} c^a \right)$$

in agreement with that of [16].

Let us consider an example of an abelian L -th stage reducible theory. We assume that $R_{a_{k+1}}^{a_k}$, $k = 0, \dots, L$, are constants and

$$R_{a_{k+1}}^{a_k} R_{a_{k+2}}^{a_{k+1}} = 0, \quad k = 0, \dots, L-1. \quad (49)$$

It is clear that (35) and (36) hold. One sees that (15) and (37) are satisfied by

$$M_{a_k} = 0, \quad k = 1, \dots, L+1.$$

In this case the extended action is given by

$$S = S_0 + \sum_{k=1}^{L+1} \phi_{a_{k-1}}^* R_{a_k}^{a_{k-1}} \phi^{a_k}.$$

The off-shell conditions (49) hold in abelian $(L+1)$ -form gauge theories. The extended action for these theories is described in [7].

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